



TITLE:

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AUTHOR(S):

Arai, Asao

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Perturbation Problem of Embedded Eigenvalues in Quantum Field Models and Representations of Canonical Commutation Relations

Asao Arai (新井朝雄) *

*Department of Mathematics, Hokkaido University
Sapporo 060, Japan*

e-mail: arai@math.sci.hokudai.ac.jp

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Abstract

We review a general theory of a new type of representation of the canonical commutation relations over a Hilbert space in connection with perturbation problem of embedded eigenvalues in a class of quantum field models.

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1 Introduction—physical background and motivation

As is well known, a nonrelativistic quantum particle with mass $m > 0$ moving in the d -dimensional Euclidean space \mathbf{R}^d under the influence of a scalar potential V (a real-valued Borel measurable function on \mathbf{R}^d) is described by the Schrödinger Hamiltonian

$$H_p := -\frac{\Delta}{2m} + V \quad (1.1)$$

acting in the Hilbert space $L^2(\mathbf{R}^d)$, where Δ is the d -dimensional generalized Laplacian. We assume that H_p is essentially self-adjoint and denote its closure by \bar{H}_p . Suppose that the particle can interact with a quantum field. Then one must replace the Hamiltonian H_p by another Hamiltonian H , taking into account the interaction between the particle

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and the quantum field. Indeed there are physical phenomena that can be explained only if such a consideration is made, e.g., the *Lamb shift* and the *spontaneous emission of light* in atoms (e.g., [17, Chapter 6]).

A standard description of a quantum field can be made in terms of a Fock space. To be concrete, let us consider a Bose quantum field whose one-particle states are described by a complex Hilbert space \mathcal{K} . The Hilbert space of state vectors of the quantum field may be taken to be the symmetric (boson) Fock space over \mathcal{K}

$$\mathcal{F}_s(\mathcal{K}) := \bigoplus_{n=0}^{\infty} \otimes_s^n \mathcal{K}, \quad (1.2)$$

where $\otimes_s^n \mathcal{K}$ denotes the n -fold symmetric tensor product Hilbert space of \mathcal{K} with $\otimes_s^0 \mathcal{K} := \mathbb{C}$. Then the free Hamiltonian of the quantum field (the Hamiltonian in the case where the quantum field has no interactions) is given by the second quantization operator

$$d\Gamma_{\mathcal{K}}(h) := \bigoplus_{n=0}^{\infty} h^{(n)}, \quad (1.3)$$

on $\mathcal{F}_s(\mathcal{K})$, where h is a self-adjoint operator on \mathcal{K} describing the one free boson and $h^{(n)}$ is the closure of the operator

$$\sum_{j=1}^n I \otimes \cdots \otimes I \otimes \overset{j}{h} \otimes I \cdots \otimes I$$

($h^{(0)} := 0$; the symbol I denotes identity operator) (for more details, see, e.g., [23, §VIII.10, Example 2], [16, §5.2]). A Hamiltonian H of the system of the above mentioned quantum particle interacting with the quantum field is given by the following form:

$$H := H_0 + H_I \quad (1.4)$$

acting in the tensor product Hilbert space $L^2(\mathbf{R}^d) \otimes \mathcal{F}_s(\mathcal{K})$, where

$$H_0 := \bar{H}_p \otimes I + I \otimes d\Gamma_{\mathcal{K}}(h) \quad (1.5)$$

and H_I is a symmetric operator describing an interaction between the quantum particle and the quantum field. Then an important task is to investigate the spectrum of H . But, here, we meet a difficult problem as explained below.

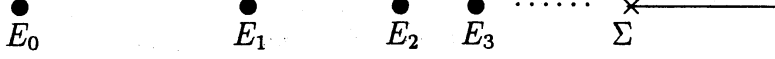
For a linear operator A on a Hilbert space, we denote its spectrum (resp. point spectrum) by $\sigma(A)$ (resp. $\sigma_p(A)$). For simplicity, suppose that the spectrum of \bar{H}_p is given as follows:

$$\begin{aligned} \sigma_p(\bar{H}_p) &= \{E_n\}_{n=0}^{\infty}, \quad E_0 < E_1 < \cdots < E_n < E_{n+1} < \cdots < \Sigma, \\ \sigma(\bar{H}_p) &= \sigma_p(\bar{H}_p) \cup [\Sigma, \infty), \end{aligned}$$

where $\Sigma \in \mathbf{R}$ is a constant.

As for h , we suppose that

$$\sigma(h) = [M, \infty), \quad \sigma_p(h) = \emptyset \quad (1.6)$$

Figure 1: The spectrum of \bar{H}_p

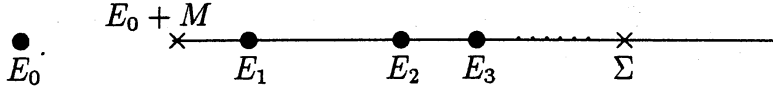
with $M \geq 0$ a constant. Then we have

$$\sigma_p(d\Gamma_K(h)) = \{0\}, \quad \sigma(d\Gamma_K(h)) = \{0\} \cup [M, \infty). \quad (1.7)$$

It follows that

$$\sigma_p(H_0) = \{E_n\}_{n=0}^\infty, \quad \sigma(H_0) = \{E_n\}_{n=0}^\infty \cup [E_0 + M, \infty). \quad (1.8)$$

This shows that all the eigenvalues E_n of H_0 with $E_n \geq E_0 + M$ are *embedded* in its continuous spectrum. In particular, if $M = 0$, then all the eigenvalues of H_0 are embedded ones. Thus to analyze the spectrum of H includes a perturbation problem of embedded eigenvalues, which are difficult to solve in general.

Figure 2: The spectrum of H_0

In the case where the quantum particle is a harmonic oscillator, i.e., V is of the form $V(x) = \mu x^2$ ($x \in \mathbf{R}^d$; $\mu > 0$ is a constant), mathematically rigorous studies on this problem have been made in a series of papers [2]–[9]. Recently more general cases and other types of models including the spin-boson model have been discussed [22], [18], [19], [20], [21], [13], [14] (see also [11], [12], [27]).

In this paper we present a brief review of the paper [10] which gives a unified approach, from a representation-theoretic point of view, to perturbation problem of embedded eigenvalues in a class of models considered in [2]–[9]. This approach is based on a new type of representation of the canonical commutation relations (CCR) over a Hilbert space and non-perturbative, making it possible to analyze exactly the spectrum of the Hamiltonian under consideration. Typical examples to which our method can be applied are as follows (the symbol \otimes for operator tensor product is omitted):

(1) *The Schwabl-Thirring model* [2, 3] .

$$H = -\frac{1}{2m}\Delta + \frac{m\omega_0^2}{2}x^2 + \int_{\mathbf{R}^d} a(k)^* a(k) \omega(k) dk + \lambda \sum_{j=1}^d x_j \phi(g_j)$$

acting in $L^2(\mathbf{R}^d) \otimes \mathcal{F}_s(L^2(\mathbf{R}^d))$, where $a(f) = \int_{\mathbf{R}^d} a(k) f(k)^* dk$, $f \in L^2(\mathbf{R}^d)$, are the annihilation operators on $\mathcal{F}_s(L^2(\mathbf{R}^d))$ (e.g., [24, §X.7], [16, §5.2]), $\phi(g_j) := (a(g_j) + a(g_j)^*)/\sqrt{2}$, $g_j \in L^2(\mathbf{R}^d)$, $\omega(k)$ is a nonnegative function denoting a dispersion relation of one boson with momentum $k \in \mathbf{R}^d$, $\omega_0 > 0$ is a constant, $\lambda \in \mathbf{R}$ is a coupling constant and $x = (x_1, \dots, x_d) \in \mathbf{R}^d$. The symbol $\int_{\mathbf{R}^d} a(k)^* a(k) \omega(k) dk$ is a formal expression of $d\Gamma_{L^2(\mathbf{R}^d)}(\omega)$.

A standard example of ω is: (i) (relativistic case) $\omega(k) = \sqrt{k^2 + M^2}$, $k \in \mathbf{R}^d$ ($M \geq 0$ is a constant); (ii) (nonrelativistic case) $\omega(k) = k^2/2M$.

(2) *The RWA model* [5] .

$$H = \sum_{j=1}^N \omega_j A_j^* A_j + \int_{\mathbf{R}^d} a(k)^* a(k) \omega(k) dk + \lambda \sum_{j=1}^N [A_j^* a(g_j) + A_j a(g_j)^*]$$

acting in $\mathcal{F}_s(\mathbf{C}^N) \otimes \mathcal{F}_s(L^2(\mathbf{R}^d))$, where each $\omega_j > 0$ is a constant and $A(z) := \sum_{j=1}^N A_j z_j^*$, $z = (z_1, \dots, z_N) \in \mathbf{C}^N$, are the annihilation operators on $\mathcal{F}_s(\mathbf{C}^N)$: $[A_j, A_k^*] = \delta_{jk}$, $[A_j, A_k] = 0$.

(3) *A generalized Schwabl-Thirring model* .

$$H = \frac{1}{2m} \sum_{j=1}^d (-iD_j - \alpha x_j)^2 + \int_{\mathbf{R}^d} a(k)^* a(k) \omega(k) dk + \lambda \sum_{j=1}^d x_j \phi(g_j)$$

acting in $L^2(\mathbf{R}^d) \otimes \mathcal{F}_s(L^2(\mathbf{R}^d))$, where D_j is the generalized partial differential operator in x_j and $\alpha \in \mathbf{R}$ is a constant.

(4) *The Pauli-Fierz model in the dipole approximation* [1, 4, 9] (see also [27])

$$H = \frac{1}{2m} \sum_{j=1}^d (-iD_j - q A_j(\varrho))^2 + \frac{m\omega_0^2}{2} x^2 + \sum_{r=1}^{d-1} \int_{\mathbf{R}^d} a_r(k)^* a_r(k) \omega(k) dk,$$

acting in $L^2(\mathbf{R}^d) \otimes \mathcal{F}_s(\oplus_{r=1}^{d-1} L^2(\mathbf{R}^d))$, where $q \in \mathbf{R}$ is a constant denoting the electric charge of the particle and $A(\varrho) = (A_1(\varrho), \dots, A_d(\varrho))$ is the quantized radiation field on $\mathcal{F}_s(\oplus_{r=1}^{d-1} L^2(\mathbf{R}^d))$ smeared out by a function ϱ with suitable regularity.

For other models, see [6] and references therein.

A basic observation for our method is in the fact that we have a natural identification

$$L^2(\mathbf{R}^d) = \mathcal{F}_s(\mathbf{C}^d),$$

so that

$$L^2(\mathbf{R}^d) \otimes \mathcal{F}_s(\mathcal{K}) = \mathcal{F}_s(\mathbf{C}^d) \otimes \mathcal{F}_s(\mathcal{K}) = \mathcal{F}_s(\mathbf{C}^d \oplus \mathcal{K})$$

Thus the quantum system consisting of a particle and a quantum field may be described in terms of *one* (extended) quantum field whose one-particle Hilbert space is $\mathbf{C}^d \oplus \mathcal{K}$. With this observation, we consider in an abstract form a quantum field theory on the Fock space $\mathcal{F}_s(\mathcal{M} \oplus \mathcal{K})$ as a representation theory of CCR (\mathcal{M} is a Hilbert space).

2 A new type of representation of the CCR over a Hilbert space

For a linear operator A on a Hilbert space, we denote its domain by $D(A)$.

Let \mathcal{H} be a complex Hilbert space with inner product $(\cdot, \cdot)_{\mathcal{H}}$ (complex linear in the second variable) and norm $\|\cdot\|_{\mathcal{H}}$. We denote by $\text{CCR}(\mathcal{H})$ the abstract $*$ -algebra (with unit element I) generated by elements $a(f), a(f)^*$ ($f \in \mathcal{H}$) satisfying the CCR over \mathcal{H}

$$[a(f), a(g)^*] = (f, g)_{\mathcal{H}} I, \quad [a(f), a(g)] = 0 = [a(f)^*, a(g)^*], \quad f, g \in \mathcal{H}, \quad (2.1)$$

with the property that the mapping $a : f \rightarrow a(f)$ from \mathcal{H} to $\text{CCR}(\mathcal{H})$ is anti-linear, where $[A, B] := AB - BA$.

Definition 2.1 A triple $\{\mathcal{F}, \mathcal{D}, \{a(f)|f \in \mathcal{H}\}\}$ consisting of a complex Hilbert space \mathcal{F} , a dense subspace \mathcal{D} of \mathcal{F} and an anti-linear mapping $a : f \rightarrow a(f)$ from \mathcal{H} to the set of closed linear operators on \mathcal{F} is called a representation of $\text{CCR}(\mathcal{H})$ if the following (i) and (ii) hold: (i) $\mathcal{D} \subset \cap_{f \in \mathcal{H}} D(a(f)) \cap D(a(f)^*)$, $a(f)\mathcal{D} \subset \mathcal{D}$, $a(f)^*\mathcal{D} \subset \mathcal{D}$ for all $f \in \mathcal{H}$; (ii) $\{a(f)|f \in \mathcal{H}\}$ fulfil the CCR (2.1) on \mathcal{D} .

A standard example of representation of $\text{CCR}(\mathcal{H})$ is given as follows. Let $\mathcal{F}_s(\mathcal{H})$ be the symmetric Fock space over \mathcal{H} . We denote by $\Omega_{\mathcal{H}} := \{1, 0, 0, \dots\}$ the Fock vacuum in $\mathcal{F}_s(\mathcal{H})$ and by $a_{\mathcal{H}}(f)$, $f \in \mathcal{H}$, the annihilation operators on $\mathcal{F}_s(\mathcal{H})$ (anti-linear in f) (e.g., [24, §X.7], [16, §5.2]). Let

$$\mathcal{F}_{\text{fin}}(\mathcal{H}) := \mathcal{L}\{\Omega_{\mathcal{H}}, a_{\mathcal{H}}(f_1)^* \cdots a_{\mathcal{H}}(f_n)^* \Omega_{\mathcal{H}} | n \geq 1, f_j \in \mathcal{H}, j = 1, \dots, n\}, \quad (2.2)$$

where $\mathcal{L}\{\dots\}$ denotes the subspace algebraically spanned by the vectors in the set $\{\dots\}$. Then $\mathcal{F}_{\text{fin}}(\mathcal{H})$ is dense and $\{\mathcal{F}_s(\mathcal{H}), \mathcal{F}_{\text{fin}}(\mathcal{H}), \{a_{\mathcal{H}}(f)|f \in \mathcal{H}\}\}$ is a representation of $\text{CCR}(\mathcal{H})$. This representation is called the *Fock representation* of $\text{CCR}(\mathcal{H})$.

As is explained in the Introduction, we are concerned with the case where \mathcal{H} is given by the direct sum of two Hilbert spaces \mathcal{M} and \mathcal{K} with $\mathcal{M} \neq \{0\}$ and $\mathcal{K} \neq \{0\}$:

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{K} = \{(v, u) | v \in \mathcal{M}, u \in \mathcal{K}\}. \quad (2.3)$$

Then we have the natural identification

$$\mathcal{F}_s(\mathcal{H}) = \mathcal{F}_s(\mathcal{M}) \otimes \mathcal{F}_s(\mathcal{K}). \quad (2.4)$$

Remark 2.2 In applications to models of a quantum particle coupled to a quantum field, the Hilbert spaces \mathcal{M} and \mathcal{K} are taken as $\mathcal{M} = \mathbb{C}^N$, $\mathcal{K} = \oplus^m L^2(\mathbb{R}^d)$ with $d, m, N \in \mathbb{N}$. Then we have

$$\mathcal{F}_s(\mathcal{H}) = \mathcal{F}_s(\mathbb{C}^N) \otimes \mathcal{F}_s(\oplus^m L^2(\mathbb{R}^d)) = L^2(\mathbb{R}^N) \otimes \mathcal{F}_s(\oplus^m L^2(\mathbb{R}^d))$$

Let $J_{\mathcal{M}}$ and $J_{\mathcal{K}}$ be conjugations on \mathcal{M} and \mathcal{K} respectively and define

$$J_{\mathcal{H}} := J_{\mathcal{M}} \oplus J_{\mathcal{K}}, \quad (2.5)$$

which is a conjugation on \mathcal{H} . For a linear operator A on \mathcal{H} and $f \in \mathcal{H}$, we set

$$A_c := J_{\mathcal{H}} A J_{\mathcal{H}}, \quad \bar{f} := J_{\mathcal{H}} f. \quad (2.6)$$

For two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, we denote by $B(\mathcal{H}_1, \mathcal{H}_2)$ the space of bounded linear operators from \mathcal{H}_1 to \mathcal{H}_2 and set $B(\mathcal{H}_1) := B(\mathcal{H}_1, \mathcal{H}_1)$.

Let S and T be elements in $B(\mathcal{K}, \mathcal{H})$ which satisfy

$$S^* S - T^* T = I_{\mathcal{K}}, \quad S^* T_c - T^* S_c = 0, \quad (2.7)$$

where $I_{\mathcal{K}}$ denotes the identity operator on \mathcal{K} .

We denote by N_b the number operator on $\mathcal{F}_s(\mathcal{H})$ ([24, §X.7], [16, §5.2]). It is well known [16, §5.2] that, for all $f \in \mathcal{H}$, $D(N_b^{1/2}) \subset D(a(f)^{\#})$ and

$$\|a(f)^{\#} \Psi\| \leq \|f\|_{\mathcal{H}} (N_b + 1)^{1/2} \|\Psi\|, \quad \Psi \in D(N_b^{1/2}), \quad (2.8)$$

where $a(f)^{\#}$ denotes either $a(f)$ or $a(f)^*$. For each $u \in \mathcal{K}$, we define an operator $b(u)$ acting in $\mathcal{F}_s(\mathcal{H})$ by

$$b(u) := a_{\mathcal{H}}(Su) + a_{\mathcal{H}}(T_c \bar{u})^*. \quad (2.9)$$

with $D(b(u)) = D(N_b^{1/2})$. It follows that $D(N_b^{1/2}) \subset D(b(u)^*)$ for all $u \in \mathcal{K}$. Hence $b(u)$ is closable. We denote its closure by the same symbol $b(u)$, so that $D(N_b^{1/2}) \subset D(b(u))$. We have

$$b(u)^* = a_{\mathcal{H}}(Su)^* + a_{\mathcal{H}}(T_c \bar{u}) \quad (2.10)$$

on $D(N_b^{1/2})$. The following fact can be easily proved.

Proposition 2.3 *The triple*

$$\pi_b := \{\mathcal{F}_s(\mathcal{H}), \mathcal{F}_{\text{fin}}(\mathcal{H}), \{b(u) | u \in \mathcal{K}\}\} \quad (2.11)$$

is a representation of $\text{CCR}(\mathcal{K})$.

The representation π_b is a basic object playing an important role in our theory.

Remark 2.4 Under the identification (2.4), we can identify $a_{\mathcal{H}}(f)^{\#}$, $f = (v, u) \in \mathcal{H}$, as

$$a_{\mathcal{H}}(f)^{\#} = a_{\mathcal{M}}(v)^{\#} \otimes I_{\mathcal{F}_s(\mathcal{K})} + I_{\mathcal{F}_s(\mathcal{M})} \otimes a_{\mathcal{K}}(u)^{\#} \quad (2.12)$$

on $\mathcal{F}_{\text{fin}}(\mathcal{M}) \otimes_{\text{alg}} \mathcal{F}_{\text{fin}}(\mathcal{K})$, where \otimes_{alg} denotes algebraic tensor product. Then there exist operators $W, V \in B(\mathcal{K})$ and $P, Q \in B(\mathcal{K}, \mathcal{M})$ such that

$$Su = (Qu, Wu), \quad Tu = (Pu, Vu), \quad u \in \mathcal{K}. \quad (2.13)$$

The operators W and Q (resp. V and P) are uniquely determined by S (resp. T). Hence we have

$$\begin{aligned} b(u) &= a_{\mathcal{M}}(Qu) \otimes I_{\mathcal{F}_s(\mathcal{K})} + I_{\mathcal{F}_s(\mathcal{M})} \otimes a_{\mathcal{K}}(Wu) \\ &\quad + a_{\mathcal{M}}(P_c \bar{u})^* \otimes I_{\mathcal{F}_s(\mathcal{K})} + I_{\mathcal{F}_s(\mathcal{M})} \otimes a_{\mathcal{K}}(V_c \bar{u})^* \end{aligned} \quad (2.14)$$

on $\mathcal{F}_{\text{fin}}(\mathcal{M}) \otimes_{\text{alg}} \mathcal{F}_{\text{fin}}(\mathcal{K})$. This is the original form of operators of the type $b(u)$ [8].

Remark 2.5 The triple $\{\mathcal{F}_s(\mathcal{H}), \mathcal{F}_{\text{fin}}(\mathcal{H}), \{a_{\mathcal{H}}(0, u) | u \in \mathcal{K}\}\}$ is a representation of $\text{CCR}(\mathcal{K})$. But this representation is not equivalent in general to the representation π_b (see Theorem 4.4 in §4 below).

Remark 2.6 The mapping $a_{\mathcal{H}}(0, \cdot) \rightarrow b(\cdot)$ may be regarded as a Bogoliubov transformation in the Fock space $\mathcal{F}_s(\mathcal{H})$. But this is a *different* type of Bogoliubov transformations from the usual ones as discussed in, e.g., [15], [25, 26].

Under additional conditions, one can express $a_{\mathcal{H}}(\cdot)$ in terms of $b(\cdot)$ and $b(\cdot)^*$:

Proposition 2.7 Suppose that S and T satisfy, in addition to (2.7),

$$SS^* - T_c T_c^* = I_{\mathcal{H}}, \quad T_c S_c^* - ST^* = 0. \quad (2.15)$$

Then, for all $f \in \mathcal{H}$,

$$a_{\mathcal{H}}(f) = b(S^* f) - b(T^* \bar{f})^*, \quad a_{\mathcal{H}}(f)^* = b(S^* f)^* - b(T^* \bar{f}). \quad (2.16)$$

on $D(N_b^{1/2})$.

Let

$$\phi_{\mathcal{H}}(f) := \frac{1}{\sqrt{2}}(a_{\mathcal{H}}(f) + a_{\mathcal{H}}(f)^*), \quad f \in \mathcal{H}, \quad (2.17)$$

which are called the Segal field operators and essentially self-adjoint on $\mathcal{F}_{\text{fin}}(\mathcal{H})$ [24, Theorem X.41]. We denote the closure of $\phi_{\mathcal{H}}(f)$ by $\overline{\phi_{\mathcal{H}}(f)}$.

An analogue of the Segal field operator is defined in the representation π_b :

$$\Phi(u) := \frac{1}{\sqrt{2}}(b(u) + b(u)^*), \quad u \in \mathcal{K}. \quad (2.18)$$

It can be proved [10] that $\Phi(u)$ is essentially self-adjoint on $\mathcal{F}_{\text{fin}}(\mathcal{H})$ and

$$\overline{\Phi(u)} = \overline{\phi_{\mathcal{H}}(Su + T_c \bar{u})}, \quad u \in \mathcal{K}. \quad (2.19)$$

We set

$$C^\infty(N_b) := \cap_{k=1}^\infty D(N_b^k). \quad (2.20)$$

Then, for all $f \in \mathcal{H}$, $a_{\mathcal{H}}(f)^\#$ leaves $C^\infty(N_b)$ invariant and so does $b(u)^\#$ for all $u \in \mathcal{K}$.

We denote by $\mathcal{I}_2(\mathcal{K}, \mathcal{H})$ the space of Hilbert-Schmidt operators from \mathcal{K} to \mathcal{H} .

Definition 2.8 Let $S, T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. We say that the pair $\langle S, T \rangle$ is in the set $\mathcal{S}(\mathcal{K}, \mathcal{H})$ if S and T satisfy (2.7), (2.15) and $T \in \mathcal{I}_2(\mathcal{K}, \mathcal{H})$.

The fundamental properties of the representation π_b are summarized in the following theorem.

Theorem 2.9 [10, Theorem 2.5]. Let $\langle S, T \rangle \in \mathcal{S}(\mathcal{K}, \mathcal{H})$. Then there exist a unit vector $\Psi_0 \in \mathcal{F}_s(\mathcal{H})$ and a unitary transformation $U : \mathcal{F}_s(\mathcal{H}) \rightarrow \mathcal{F}_s(\mathcal{K})$ such that the following (a)–(d) hold:

- (a) $\Psi_0 \in C^\infty(N_b)$ and, for all $u \in \mathcal{K}$, $b(u)\Psi_0 = 0$.
- (b) The subspace $\mathcal{L}\{\Psi_0, b(u_1)^* \cdots b(u_n)^* \Psi_0 | n \geq 1, u_j \in \mathcal{K}, j = 1, \dots, n\}$ is dense in $\mathcal{F}_s(\mathcal{H})$.
- (c) $U\Psi_0 = \Omega_{\mathcal{K}}$ and $Ub(u_1)^* \cdots b(u_n)^* \Psi_0 = a_{\mathcal{K}}(u_1)^* \cdots a_{\mathcal{K}}(u_n)^* \Omega_{\mathcal{K}}$ for all $n \geq 1, u_j \in \mathcal{K}, j = 1, \dots, n$.
- (d) For all $u \in \mathcal{K}$,

$$U\overline{\Phi(u)}U^{-1} = \overline{\phi_{\mathcal{K}}(u)}, \quad Ub(u)^*U^{-1} = a_{\mathcal{K}}(u)^*.$$

Moreover, Ψ_0 is the only one (up to scalar multiples) of vectors Ψ such that $\Psi \in D(N_b^{1/2})$ and $b(u)\Psi = 0$ for all $u \in \mathcal{K}$.

3 Construction of a Hamiltonian

By using the representation π_b given by (2.11), we can construct a self-adjoint Hamiltonian acting in $\mathcal{F}_s(\mathcal{H})$ whose spectrum can be exactly identified. In application to perturbation problem of embedded eigenvalues in quantum systems of quantum particles interacting with quantum fields, this class of Hamiltonians gives a class of exactly solvable models [7, 8].

For every $K \in \mathcal{I}_2(\mathcal{H}) := \mathcal{I}_2(\mathcal{H}, \mathcal{H})$, there exist (not necessarily complete) orthonormal sets $\{\psi_n\}_{n=1}^M$ and $\{\phi_n\}_{n=1}^M$ in \mathcal{H} (M may be finite or infinite) and positive real numbers $\{\lambda_n\}_{n=1}^M$ such that $\sum_{n=1}^M \lambda_n^2 < \infty$,

$$K = \sum_{n=1}^M \lambda_n (\psi_n, \cdot) \phi_n, \quad (3.1)$$

where, in the case $M = \infty$, the sum in (3.1) converges in operator norm (e.g., [23, Theorem VI.17, Theorem VI.22]). We define for a finite positive integer N

$$\langle a_{\mathcal{H}}^* | K_N | a_{\mathcal{H}}^* \rangle = \sum_{n=1}^{\min\{M, N\}} \lambda_n a_{\mathcal{H}}(\bar{\psi}_n)^* a_{\mathcal{H}}(\phi_n)^* \quad (3.2)$$

and

$$\langle a_{\mathcal{H}} | K_N | a_{\mathcal{H}} \rangle = \sum_{n=1}^{\min\{M, N\}} \lambda_n a_{\mathcal{H}}(\psi_n) a_{\mathcal{H}}(\bar{\phi}_n). \quad (3.3)$$

Then we can show that, for all $\Psi \in \mathcal{F}_{\text{fin}}(\mathcal{H})$, the strong limits

$$\langle a_{\mathcal{H}}^* | K | a_{\mathcal{H}}^* \rangle \Psi := s\text{-}\lim_{N \rightarrow \infty} \langle a_{\mathcal{H}}^* | K_N | a_{\mathcal{H}}^* \rangle \Psi \quad (3.4)$$

and

$$\langle a_{\mathcal{H}} | K | a_{\mathcal{H}} \rangle \Psi := s\text{-}\lim_{N \rightarrow \infty} \langle a_{\mathcal{H}} | K_N | a_{\mathcal{H}} \rangle \Psi \quad (3.5)$$

exist. Moreover, the operator $\langle a_{\mathcal{H}}^{\#} | K | a_{\mathcal{H}}^{\#} \rangle$ defined on $\mathcal{F}_{\text{fin}}(\mathcal{H})$ is closable and

$$\langle a_{\mathcal{H}}^* | K | a_{\mathcal{H}}^* \rangle^* = \langle a_{\mathcal{H}} | K^* | a_{\mathcal{H}} \rangle \quad (3.6)$$

on $\mathcal{F}_{\text{fin}}(\mathcal{H})$. We denote the closure of $\langle a_{\mathcal{H}}^{\#}|K|a_{\mathcal{H}}^{\#}\rangle$ by the same symbol.

For a densely defined closed linear operator A on \mathcal{H} , we denote by $d\Gamma_{\mathcal{H}}(A)$ the second quantization operator on $\mathcal{F}_s(\mathcal{H})$ [23, p.302, Example 2], which is the closed linear operator on $\mathcal{F}_s(\mathcal{H})$ such that $d\Gamma_{\mathcal{H}}(A)\Omega_{\mathcal{H}} = 0$ and

$$d\Gamma_{\mathcal{H}}(A)a_{\mathcal{H}}(f_1)^* \cdots a_{\mathcal{H}}(f_n)^*\Omega_{\mathcal{H}} = \sum_{j=1}^n a_{\mathcal{H}}(f_1)^* \cdots a_{\mathcal{H}}(Af_j)^* \cdots a_{\mathcal{H}}(f_n)^*\Omega_{\mathcal{H}},$$

for all $f_1, \dots, f_n \in D(A)$ and $n \geq 1$.

Let $\langle S, T \rangle \in \mathcal{S}(\mathcal{K}, \mathcal{H})$ and h be a nonnegative self-adjoint operator on \mathcal{K} such that $h = h_c$ and the following properties (h.1)–(h.3) hold:

(h.1) The subspace $\mathcal{H}_0 := \{f \in \mathcal{H} | S^*f, T_c^*f \in D(h)\}$ is dense in \mathcal{H} .

(h.2) ThS^* and $Th^{1/2}$ respectively define a Hilbert-Schmidt operator on \mathcal{H} and from \mathcal{K} to \mathcal{H} .

(h.3) The subspace $D_S(h) := \{u \in D(h) | S^*Su \in D(h)\}$ is a core of h .

It follows that $ShS^* + T_chT_c^*$ is densely defined, hence a symmetric operator on \mathcal{H} and $D(ShT^*)$ is dense and defines a Hilbert-Schmidt operator on \mathcal{H} .

We define

$$H := d\Gamma_{\mathcal{H}}(\overline{ShS^* + T_chT_c^*}) + \langle a_{\mathcal{H}} | \overline{ThS^*} | a_{\mathcal{H}} \rangle + \langle a_{\mathcal{H}} | \overline{ThS^*} | a_{\mathcal{H}} \rangle^*, \quad (3.7)$$

and set

$$E := -\|Th^{1/2}\|_{\text{HS}}^2, \quad (3.8)$$

where $\|\cdot\|_{\text{HS}}$ denotes Hilbert-Schmidt norm. The operator H gives an abstract form unifying Hamiltonians of models of a quantum harmonic oscillator coupled to a quantized field [2]–[9] (see the Introduction).

Let

$$\mathcal{F}_{\text{fin}}(\mathcal{H}_0) = \mathcal{L}\{\Omega_{\mathcal{H}}, a_{\mathcal{H}}(f_1)^* \cdots a_{\mathcal{H}}(f_n)^*\Omega_{\mathcal{H}} | n \geq 1, f_j \in \mathcal{H}_0, j = 1, \dots, n\} \quad (3.9)$$

Obviously $\mathcal{F}_{\text{fin}}(\mathcal{H}_0) \subset D(H)$. Hence H is a symmetric operator. We can prove the following fact.

Theorem 3.1 [10, Theorem 3.1]. *The operator H is essentially self-adjoint on $\mathcal{F}_{\text{fin}}(\mathcal{H}_0)$ and its closure \bar{H} is unitarily equivalent to $d\Gamma_{\mathcal{K}}(h) + E$ under the unitary transformation U given in Theorem 2.9: $U\bar{H}U^{-1} = d\Gamma_{\mathcal{K}}(h) + E$.*

As a corollary to Theorem 3.1, we can identify the spectrum of \bar{H} :

Corollary 3.2

$$\begin{aligned} \sigma(\bar{H}) &= \sigma(d\Gamma_{\mathcal{K}}(h) + E), & \sigma_{\text{ac}}(\bar{H}) &= \sigma_{\text{ac}}(d\Gamma_{\mathcal{K}}(h) + E), \\ \sigma_s(\bar{H}) &= \sigma_s(d\Gamma_{\mathcal{K}}(h) + E), & \sigma_p(\bar{H}) &= \sigma_p(d\Gamma_{\mathcal{K}}(h) + E), \end{aligned}$$

where σ_s and σ_{ac} denote singular continuous spectrum and absolutely continuous spectrum respectively. The multiplicity of each eigenvalue of \bar{H} is the same as that of the corresponding one of $d\Gamma_{\mathcal{K}}(h) + E$. In particular, \bar{H} has a unique ground state given by the vector Ψ_0 (up to constant multiples) with the ground state energy E .

In concrete models, the unperturbed Hamiltonian H_0 is of the form

$$H_0 = d\Gamma_{\mathcal{H}}(\ell \oplus h) = d\Gamma_{\mathcal{M}}(\ell) \otimes I_{\mathcal{F}_s(\mathcal{K})} + I_{\mathcal{F}_s(\mathcal{M})} \otimes d\Gamma_{\mathcal{K}}(h), \quad (3.10)$$

where ℓ is a self-adjoint operator on \mathcal{M} bounded from below (see the examples given in the Introduction). We write

$$H = H_0 + H_I \quad (3.11)$$

with

$$H_I = d\Gamma_{\mathcal{H}}(\overline{ShS^*} + T_chT_c^*) - d\Gamma_{\mathcal{H}}(\ell \oplus h) + \langle a_{\mathcal{H}} | \overline{ThS^*} | a_{\mathcal{H}} \rangle + \langle a_{\mathcal{H}} | \overline{ThS^*} | a_{\mathcal{H}} \rangle^*. \quad (3.12)$$

For this form of H , Corollary 3.2 implies the following. For simplicity, consider the case where $\sigma(h)$ is purely continuous as is given by (1.6) and $\sigma(\ell)$ is purely discrete so that

$$\sigma(d\Gamma_{\mathcal{M}}(\ell)) = \sigma_p(d\Gamma_{\mathcal{M}}(\ell)) = \{E_n\}_{n=0}^{\infty}$$

with $E_0 < E_1 < E_2 < \dots$ (E_n is determined by $\sigma(\ell)$). Then we have (1.7) and hence (1.8). Thus each E_n is an eigenvalue of H_0 and the eigenvalues $E_n \geq E_0 + M$ are embedded in the continuous spectrum of H_0 . On the other hand, Corollary 3.2 implies that

$$\sigma(\bar{H}) = \{E\} \cup [E + M, \infty), \quad \sigma_p(\bar{H}) = \{E\}.$$

Hence all the embedded eigenvalues $E_n \geq E_0 + M$ turn out to disappear under the perturbation H_I , i.e., they are unstable under the perturbation H_I (we may regard $E_n < E_0 + M$ as eigenvalues changing to E or $E + M$ under the perturbation H_I). Thus \bar{H} gives, in an abstract form, a class of self-adjoint operators acting in the Fock space $\mathcal{F}_s(\mathcal{H})$, which describe the instability phenomenon of embedded eigenvalues.

4 Structure of the representation π_b

We write each vector $f \in \mathcal{H}$ as

$$f = (f_{\mathcal{M}}, f_{\mathcal{K}}), \quad f_{\mathcal{M}} \in \mathcal{M}, f_{\mathcal{K}} \in \mathcal{K}.$$

For $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, we define $\tilde{A} \in \mathcal{B}(\mathcal{H})$ by

$$\tilde{A}f := Af_{\mathcal{K}}, \quad f \in \mathcal{H}. \quad (4.1)$$

Then we have

$$\tilde{A}^*f = (0, A^*f), \quad f \in \mathcal{H}. \quad (4.2)$$

It is easy to show that, for all $A, B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$,

$$\tilde{A}\tilde{B}^* = AB^*, \quad \tilde{B}^*\tilde{A}f = (0, B^*Af_{\mathcal{K}}), f \in \mathcal{H}. \quad (4.3)$$

Let $\langle S, T \rangle \in \mathcal{S}(\mathcal{K}, \mathcal{H})$ and $P_{\mathcal{K}}$ be the orthogonal projection from \mathcal{H} onto \mathcal{K} . Then we have

$$\tilde{S}^*\tilde{S} - \tilde{T}^*\tilde{T} = P_{\mathcal{K}}, \quad \tilde{S}^*\tilde{T}_c - \tilde{T}^*\tilde{S}_c = 0, \quad (4.4)$$

$$\tilde{S}\tilde{S}^* - \tilde{T}_c\tilde{T}_c^* = I_{\mathcal{H}}, \quad \tilde{T}_c\tilde{S}_c^* - \tilde{S}\tilde{T}^* = 0. \quad (4.5)$$

Let $L \in \mathcal{B}(\mathcal{H})$ be such that

$$L^*L = P_{\mathcal{K}}, \quad LL^* = I_{\mathcal{H}}. \quad (4.6)$$

Then L is a partial isometry on \mathcal{H} with initial space \mathcal{K} and final space \mathcal{H} .

We define $X, Y \in \mathcal{B}(\mathcal{H})$ by

$$X := \tilde{S}L^*, \quad Y := \tilde{T}L^*.$$

Then one can prove the following fact.

Lemma 4.1 [10, Lemma 4.1]. *The following relations hold:*

$$X^*X - Y^*Y = I_{\mathcal{H}}, \quad X^*Y_c - Y^*X_c = 0, \quad (4.7)$$

$$XX^* - Y_cY_c^* = I_{\mathcal{H}}, \quad Y_cX_c^* - XY^* = 0. \quad (4.8)$$

Moreover, $Y \in \mathcal{I}_2(\mathcal{H})$.

For each $f \in \mathcal{H}$, we define an operator $c(f)$ by

$$c(f) := a_{\mathcal{H}}(Xf) + a_{\mathcal{H}}(Y_c\bar{f})^* \quad (4.9)$$

with $D(c(f)) = D(N_b^{1/2})$. Then $c(f)$ is closable. We denote its closure by the same symbol.

Theorem 4.2 [10, Theorem 4.2]. *The mapping $\{a_{\mathcal{H}}, a_{\mathcal{H}}^*\} \rightarrow \{c, c^*\}$ is a proper Bogoliubov (canonical) transformation on $\mathcal{F}_s(\mathcal{H})$, i.e., there exists a unitary operator $U_{\mathcal{H}}$ on $\mathcal{F}_s(\mathcal{H})$ such that, for all $f \in \mathcal{H}$,*

$$c(f) = U_{\mathcal{H}}a_{\mathcal{H}}(f)U_{\mathcal{H}}^{-1}, \quad c(f)^* = U_{\mathcal{H}}a_{\mathcal{H}}(f)^*U_{\mathcal{H}}^{-1}$$

As a corollary to Theorem 4.2, we have the following.

Corollary 4.3 *For all $u \in \mathcal{K}$,*

$$b(u) = U_{\mathcal{H}}a_{\mathcal{H}}(L(0, u))U_{\mathcal{H}}^{-1}, \quad b(u)^* = U_{\mathcal{H}}a_{\mathcal{H}}(L(0, u))^*U_{\mathcal{H}}^{-1}. \quad (4.10)$$

We next consider expressing $a_{\mathcal{H}}(L(0, \cdot))$ as a transformation of $a_{\mathcal{H}}(0, \cdot)$.

Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces and $C \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ be a contraction operator, i.e., $\|C\| \leq 1$. Then we can define a contraction operator $\Gamma_{\mathcal{H}_1, \mathcal{H}_2}(C) : \mathcal{F}_s(\mathcal{H}_1) \rightarrow \mathcal{F}_s(\mathcal{H}_2)$ by

$$\Gamma_{\mathcal{H}_1, \mathcal{H}_2}(C) := \bigoplus_{n=0}^{\infty} (\otimes^n C) \quad (4.11)$$

with $\otimes^0 C := 1$, where $\otimes^n C$ denotes the n -fold tensor product of C .

In the case where C is a contraction operator on a single Hilbert space \mathcal{H}_1 , we set

$$\Gamma_{\mathcal{H}_1}(C) := \Gamma_{\mathcal{H}_1, \mathcal{H}_1}(C). \quad (4.12)$$

We have

$$\Gamma_{\mathcal{H}}(L)\Gamma_{\mathcal{H}}(L)^* = I_{\mathcal{F}_s(\mathcal{H})}, \quad \Gamma_{\mathcal{H}}(L)^*\Gamma_{\mathcal{H}}(L) = \Gamma_{\mathcal{H}}(P_{\mathcal{K}}). \quad (4.13)$$

It is easy to see that $\Gamma_{\mathcal{H}}(P_{\mathcal{K}})$ is the orthogonal projection onto the closed subspace $\mathcal{F}_s(\{0\} \oplus \mathcal{K}) = \mathbb{C} \otimes \mathcal{F}_s(\mathcal{K})$. Hence $\Gamma_{\mathcal{H}}(L)$ is a partial isometry on $\mathcal{F}_s(\mathcal{H})$. Let

$$V_{\mathcal{H}} := U_{\mathcal{H}} \Gamma_{\mathcal{H}}(L). \quad (4.14)$$

Then

$$V_{\mathcal{H}} V_{\mathcal{H}}^* = I_{\mathcal{F}_s(\mathcal{H})}, \quad V_{\mathcal{H}}^* V_{\mathcal{H}} = \Gamma_{\mathcal{H}}(P_{\mathcal{K}}), \quad (4.15)$$

which imply that $V_{\mathcal{H}}$ is a partial isometry on $\mathcal{F}_s(\mathcal{H})$ with initial space $\mathbb{C} \otimes \mathcal{F}_s(\mathcal{K})$ and final space $\mathcal{F}_s(\mathcal{H})$. We can prove the following fact..

Theorem 4.4 [10, Corollary 4.5]. *For all $u \in \mathcal{K}$,*

$$b(u) = V_{\mathcal{H}} a_{\mathcal{H}}(0, u) V_{\mathcal{H}}^*, \quad b(u)^* = V_{\mathcal{H}} a_{\mathcal{H}}(0, u)^* V_{\mathcal{H}}^*. \quad (4.16)$$

This theorem shows that the representation π_b is a transformation of the representation $\{a_{\mathcal{H}}(0, u) | u \in \mathcal{K}\}$ by the *partial isometry* $V_{\mathcal{H}}$, which is a composition of the partial isometry $\Gamma_{\mathcal{H}}(L)$ and the proper Bogoliubov transformation $U_{\mathcal{H}}$.

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